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LETTER TO THE EDITOR

Painlevé analysis and singularity confinement: the ultimate conjecture

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Abstract. We propose a new integrability criterion based on the combination of two powerful integrability detectors: the 'Painlevé method' (for continuous systems) and the singularity confinement (for discrete systems). With this new criterion we can treat a large variety of systems that combine discrete and continuous characteristics: differential-difference systems (ordinary and partial), differential-delay problems and even a class of integro-differential equations.

The Painlevé method, based on the singularity analysis of the solutions of nonlinear differential equations, has turned out to be the most reliable integrability detector over the past decade [1,2]. On the other hand, the ever increasing interest in integrable mappings and lattices has recently spurred the investigations that resulted in the proposal of an integrability detector for discrete systems: the so-called singularity confinement [3]. Although for both methods the key word is singularity, their formulation (in fact their very essence) was such that neither could be adapted to systems that are genuinely *both* discrete and continuous. Clearly, a synthesis was needed. In what follows, we will propose such a method, incorporating features of both approaches, and which will allow us to treat a large variety of systems. Our conjecture will thus provide an integrability criterion for discrete–continuous systems [4].

The singularity confinement method has been put forward as an integrability test for purely discrete systems. In the year that followed its appearance a host of results have been obtained confirming its predictive value and its usefulness. The essence of this method is quite simple. Consider a mapping (for the simplicity of the presentation we limit ourselves here to one-dimensional systems) in which a given term is expressed as a function of a (small) number of the preceding ones. Now it may well happen that, due to particular initial conditions, this term diverges. The consequences of this infinite value will propagate with the iteration of the mapping and thus the subsequent terms may diverge (or vanish, or in any other way behave abnormally) as well. Our conjecture, based on the observed behaviour, is that when the mapping is integrable, these divergences or anomalies do not propagate *ad infinitum* but rather disappear after a few steps. We then say that the singularity is confined. The divergence usually appears because of the vanishing of some denominator. One can introduce an expansion parameter by assuming that this denominator becomes very small but does not quite vanish and accordingly expand the solution of the mapping locally.

On the other hand, the, by now, classical singularity analysis, or 'Painlevé method' has become a standard tool for the investigation of the integrability of continuous systems. The idea here is that, given a nonlinear differential equation, one looks for *singular solutions around a movable (i.e. initial condition dependent) singularity*. If all these singular solutions can be expressed as Laurent series around the singularity then we say that the equation has the Painlevé property. In order to combine the two methods, assuming that we have a differential-difference system, we will consider the effect of a singularity in the continuous variable when it is iterated following the discrete evolution. The Painlevé property requires that this singularity be pole-like and the singularity confinement criterion consists in demanding that this singularity disappears after some iterations (in the discrete variable). The procedure will hopefully become clearer with the examples we give later.

Let us start with the auto-Bäcklund relation for the potential-KdV equation [5], that allows us to relate two solutions with numbers of kinks differing by one unit

$$u'_{n+1} + u'_n + (u_{n+1} - u_n)^2 + K_n = 0. \tag{1}$$

where the prime denotes the derivative with respect to the continuous variable x, and K_n is related to the energy of the kink which is added (or removed) and should, in principle, depend on n but *not* on x. We will indeed see this fact essentially emerge from our analysis.

In the spirit of the singularity confinement method, we assume that u_n is regular and that a singularity first appears for u_{n+1}. Given the Riccati form of (1) this singularity can only be a simple pole: $u_{n+1} = 1/(x-x_0)$ with resonance $\nu = -1$ due to the arbitrariness of x_0. Given u_{n+1} we can compute u_{n+2}. The Riccati now has divergent coefficients and they will induce a singularity that, for u_{n+2}, will have a position x_0 *fixed* by u_{n+1}. We can easily prove that if u_k is of the form $u_k = m(m-1)/2(x-x_0)$ then u_{k+1} will diverge as $A/(x-x_0)$ with $A = \frac{1}{2}m(m+1)$ or $A = \frac{1}{2}(m-1)(m-2)$. Thus starting from $u_{n+1} = 1/(x-x_0)$ we get u_{n+2} that either diverges as $3/(x-x_0)$ or has a finite value. In the latter case the singularity should be confined while a resonance $\nu = 3$ exists. Thus we must expand u_{n+2} in power series and evaluate the resonance compatibility condition. It turns out to be just $\mathrm{d}K_n/\mathrm{d}x = \mathrm{d}K_{n-1}/\mathrm{d}x$, independent of n. This would lead to $K_n = \kappa_n + 2\phi'(x)$. However by redefining u: $u \to u + \phi(x)$, we can absorb ϕ entirely and without loss of generality we can assume that K_n is indeed x-independent. When this condition is satisfied u_{n+2} is regular, i.e. the singularity is indeed confined. When u_{n+2} diverges we pursue the analysis one step further: u_{n+3} diverges either as $6/(x-x_0)$ or as $1/(x-x_0)$. The first case takes us deeper into the singularity (but in such a way that we can always re-emerge eventually), while, if we follow the second, we see readily that we can emerge at the very next step. We have checked that under the sole condition: $\mathrm{d}K_n/\mathrm{d}x$ *is independent of* n, the singularity is also confined in this case. By now the pattern has become clear: a singularity appears at a certain iteration and, when iterated, it becomes confined. The process may necessitate an arbitrarily large number of steps, but from the structure of the singularity it is clear that we can always emerge. Moreover, the simplest case, i.e. confinement in one step, is the generic one. It is precisely this confinement pattern that gives the constraint on K_n: once it is satisfied, equation (1) becomes integrable. Systems such as (1) were previously treated following the 'classical' Painlevé approach, usually through some periodicity assumption. One assumed that for some N, $u_n = u_{n+N}$, and explored the integrability of the finite N-order system for increasing (but still typically small) values of N. This method has two major drawbacks. The first is that small-N cases are far from being generic. The second is more serious for non-autonomous systems. If the n-dependent coefficients (such as K_n in equation (1)) are *not* periodic, there is simply no way to perform the Painlevé analysis through periodification. In contrast, the singularity confinement approach deals with such cases in a straightforward way.

The previous case has clearly shown the interplay between the processes of (continuous) singularity analysis and confinement. The situation becomes even simpler when one considers differential-difference equations where the evolution does not enter through derivatives. We will illustrate this through the equation

$$q_n' = (aq_n^2 + bq_n + c)(q_{n+1} - q_{n-1}) \tag{2}$$

that contains both the discrete MKdV [6] ($-a = c = 1,\ b = 0$) and the Kac–Van Moerbeke system [7] ($a = c = 0,\ b = 1$), two well-known integrable equations. We start by rewriting (2) as

$$q_{n+1} = q_{n-1} + \frac{q_n'}{aq_n^2 + bq_n + c} \tag{3}$$

and assume that q_n is regular: $q_n = g + g'(x - x_0) + \frac{1}{2}g''(x - x_0)^2 + \cdots$, such that $ag^2 + bg + c = 0$. Thus q_{n+1} develops a singularity at x_0:

$$q_{n+1} = 1/(2ag + b)(x - x_0) + q_{n-1} + \cdots.$$

Iterating we find $q_{n+2} = -g - b/a + (x - x_0)A + \cdots$ where A depends on g, g' and g'' as well as on q_{n-1}. The value of q_{n+2} at $x = x_0$ is a zero of the denominator and thus potentially dangerous. However, it is precisely this value that cancels the infinite contribution of q_{n+1} in q_{n+3}. Thus the latter has a finite value and the singularity is confined. Note that in this case all singularities are, by construction, pure poles: we do not integrate any differential equation, thus there are no resonances and therefore no resonance conditions. The 'continuous' Painlevé part is *automatically* satisfied, and only the confinement is operative. Two further remarks are in order at this point. The Kac–Van Moerbeke equation is not included in this analysis since $a = 0$. However its analysis follows the same steps. We start with

$$q_n = g'(x - x_0) + \tfrac{1}{2}g''(x - x_0)^2 + \cdots$$

leading to

$$q_{n+2} = -1/(x - x_0) + (x - x_0)A + \cdots \qquad q_{n+3} = (x - x_0)B + \cdots$$

and finally a non-singular q_{n+4}, that depends on q_{n-1} and g. The second remark concerns the assumption that the first derivative of g does not vanish. Although this is the simplest case (the one we consider as 'generic') it is not the only one.

For instance in the discrete MKdV case, if $g' = 0$ then A in the expression of q_{n+2} diverges and the singularity will need extra steps to be confined. Namely

$$q_{n+1} = 2/(2ag+b)(x-x_0) + \mathcal{O}(1)$$
$$q_{n+2} = -b/2a + \mathcal{O}(x-x_0)$$
$$q_{n+3} = 2/(2ag+b)(x-x_0) + \mathcal{O}(1)$$
$$q_{n+4} = -g - b/a + \mathcal{O}((x-x_0)^2)$$

and the singularity is finally confined as q_{n+5} is regular. In principle, one must consider *all* the possible singular behaviours; vanishing derivatives of g up to arbitrary order. Our claim is that the essential constraints arise already at the lowest-order singularity.

Let us now consider the application of our technique to partial differential equations. As an example we will choose the two-dimensional Toda lattice [8] that we extend so as to make it non-autonomous. It is most convenient here to use the bilinear formalism [9] and adopt the notation $\underline{\tau} \equiv \tau_{n-1}$, $\tau \equiv \tau_n$, $\overline{\tau} \equiv \tau_{n+1}$, etc. We have

$$(\overline{\tau}\underline{\tau} - \tau^2)a(n,x,y) = \tau\tau_{xy} - \tau_x\tau_y \tag{4}$$

(the usual form corresponds to $a(n,x,y) = 1$). Solving for $\overline{\tau}$ we have

$$\overline{\tau} = \frac{1}{\underline{\tau}}\left[\frac{\tau\tau_{xy} - \tau_x\tau_y}{a(n,x,y)} + \tau^2\right]. \tag{5}$$

Next we assume that for some τ the term in square brackets vanishes at $(x,y) = (0,0)$, and thus $\overline{\tau}$ vanishes also. Using the form $\overline{\tau} = fx + gy + \mathcal{O}(x^2, xy, y^2)$ we compute $\overline{\overline{\tau}}$. Now, because of the vanishing of $\overline{\tau}$, $\overline{\overline{\overline{\tau}}}$ would diverge unless the terms in square brackets vanishes when computed for $\overline{\overline{\tau}}$ and $a(n+2,x,y)$. We implemented this condition (a calculation that can be performed in a straightforward way using the REDUCE algebraic manipulation language) and combining it with the same condition for τ and $a(n,x,y)$ we obtain a constraint for a that reads

$$\overline{a} - 2a + \underline{a} = (\log a)_{xy}. \tag{6}$$

This equation for a is nothing else but the two-dimensional Toda lattice as can readily be seen by taking $\log a = u - \underline{u}$ whereupon (6) writes $u_{xy} = \mathrm{e}^{\overline{u}-u} - \mathrm{e}^{u-\underline{u}}$. Several interesting solutions exist to (6). If we take a to be independent of (x,y), we find $a = \alpha n + \beta$, i.e. a Toda lattice with non-constant coefficients [10] that belongs to the more general family of the Hirota–Miwa equation [11]. Assuming that τ and a depend on (x,y) only through $r = \sqrt{xy}$ and a independent of n we obtain a generalized cylindrical Toda. Its usual form [12] results from the assumption $a = 1$: $(\overline{\tau}\underline{\tau} - \tau^2) = \tau(\tau_{rr} + r^{-1}\tau_r) - \tau_r{}^2$. Finally, taking $x = y$ we recover (for $a = 1$) the one-dimensional Toda lattice [13]. Putting $\tau = \mathrm{e}^w$ and taking $u = \overline{w} - w$ we indeed find

$$u'' = \mathrm{e}^{\overline{u}-u} - \mathrm{e}^{u-\underline{u}}. \tag{7}$$

Equation (7) is usually interpreted as a relation between the dynamical variables u_n that are functions of x. However, there exists another interesting interpretation. We can consider that (7) relates *one* function $u(x)$ at different 'times' $x + n\xi$ where ξ is some fixed quantity. Thus equation (7), written now as $u''(x) = \mathrm{e}^{u(x+\xi)-u(x)} - \mathrm{e}^{u(x)-u(x-\xi)}$ now becomes a differential-delay equation. It is precisely this equation that was obtained by Levi and Winternitz in [14] as reductions of the Toda lattice through a combination of discrete and continuous symmetries. As far as the singularity analysis (in the general sense presented in this article) is concerned, the non-local relation between u and $\overline{u}$, namely $\overline{u}(x) = u(x + \xi)$, does not imply any special treatment: this differential-delay equation passes the test *exactly* in the same way as the Toda lattice. On the other hand, this novel interpretation of (7) will allow us to extend our approach to genuinely non-local equations like the integro-differential intermediate long-wave (ILW) equation [15].

The ILW equation can be written in bilinear formalism [16] as

$$(\mathrm{i}D_t + \mathrm{i}h^{-1}D_x - D_x{}^2)\overline{F} \cdot F = 0 \tag{8}$$

where $F = F(x - \mathrm{i}h, t)$ and $\overline{F} = F(x + \mathrm{i}h, t)$ where h is a real parameter. For $h \to \infty$, this equation goes over to the Benjamin–Ono equation. Its essential non-locality comes from the fact that it relates the function at point $(x + \mathrm{i}h)$ to the function at $(x - \mathrm{i}h)$. Thus, the usual nonlinear form of the equation requires the introduction of integro-differential terms. Here, we will perform the analysis directly on the bilinear form (8). Using the explicit expression of the Hirota operator D we rewrite (8) as

$$\mathrm{i}(\overline{F}_t + h^{-1}\overline{F}_x)F + \lambda\mathrm{i}\overline{F}(F_t + h^{-1}F_x) - \overline{F}_{xx}F - \overline{F}F_{xx} + 2\overline{F}_xF_x = 0 \tag{9}$$

where, from (8), $\lambda = -1$. However, we will keep this parameter free in the analysis. We assume that $\underline{F}(\equiv F(x - 3\mathrm{i}h))$ is regular and that F vanishes as ϕ^m on a singularity manifold $\phi(x, t)$. Since $\underline{F}$ is regular the only possibility for F is to vanish linearly $F \sim \phi$, i.e. $m = 1$. We expand both $\underline{F}$ and F in powers of ϕ: $\underline{F} = \sum_{k=0} \underline{F}_k \phi^k$, $F = \sum_{k=1} F_k \phi^k$ and determine $F_2, F_3, F_4, \ldots$ from the equation relating F and $\underline{F}$, i.e. the 'down-shift' of (9). The resonances being -1 and 0, we have ϕ and F_1 as free functions. (In order to perform the calculations more easily we use the Kruskal ansatz [17] $\phi = x + f(t)$, but this is nothing but a practical convenience.) Next we consider the equation for $\overline{F}$. Requiring a singularity confinement we ask that $\overline{F}$ be regular and we expand $\overline{F} = \sum_{k=0} \overline{F}_k \phi^k$. The resonances of (9) are 0 and 3 and, so, $\overline{F}_0$ and $\overline{F}_3$ are free. We compute, thus, $\overline{F}_1$, $\overline{F}_2$ and substitute into the compatibility condition for the resonance (3) that reads $0 \cdot \overline{F}_3 = Q$. Using the values of the F_i we computed, we find that Q has a factor $(\lambda + 1)$. Thus compatibility is satisfied and the singularity is confined, provided $\lambda = -1$, which corresponds precisely to the exact form of the ILW equation (8).

From the examples presented we can conclude that a new, powerful method for the investigation of integrability has emerged. The usual Painlevé method (for continuous systems) related integrability to the nature of the singularities. The singularity confinement method for discrete systems requires that singularities do not propagate *ad infinitum* if the system is to be integrable. The continuous part of the system generates a movable singularity that must eventually (in the course of the discrete iterations) disappear. Thus our conjecture is that for systems that are both discrete and continuous integrability is related to confined singularities. The examples that we presented have shown precisely how one can apply the method. In principle, one must examine an arbitrarily large number of singularities. However, in practice, it turns out that the simplest singularity is the 'generic' one: the essential constraints for integrability stem from its study. So we expect in most physically interesting examples to be able to obtain sufficient evidence on integrability from the study of a small number of singularities.

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